Resummation in QCD Fractional Analytic Perturbation Theory

Alexander P. Bakulev

Bogoliubov Laboratory of Theoretical Physics
Joint Institute for Nuclear Research, Dubna, 141980 Russia

We describe the generalization of Analytic Perturbation Theory (APT) for QCD observables, initiated by Radyushkin, Krasnikov, Pivovarov, Shirkov and Solovtsov, to fractional powers of coupling — Fractional APT (FAPT). The basic aspects of FAPT is shortly summarized. We describe how to treat heavy-quark thresholds in FAPT and then show how to resum perturbative series in both the one-loop APT and FAPT. As an application we consider FAPT description of the Higgs boson decay $H^0 \to b\bar{b}$. The main conclusion is: To achieve an accuracy of the order of 1% it is enough to take into account up to the third correction.

1. APT and FAPT in QCD

In the standard QCD Perturbation Theory (PT) we know that the Renormalization Group (RG) equation $da_s[L]/dL = -a_s^2 - \ldots$ for the effective coupling $\alpha_s(Q^2) = a_s[L]/\beta_f$ with $L = \ln(Q^2/\Lambda^2)$, $\beta_f = b_0(N_f)/(4\pi) = (11-2N_f/3)/(4\pi)$[1]. Then the one-loop solution generates Landau pole singularity, $a_s[L] = 1/L$.

Strictly speaking the QCD Analytic Perturbation Theory (APT) was initiated by N. N. Bogolyuov et al. paper of 1959 [1], where ghost-free effective coupling for QED has been constructed. Then in 1982 Radyushkin [2] and Krasnikov and Pivovarov [3] using the same dispersion technique suggested regular (for $s \geq \Lambda^2$) QCD running coupling in Minkowskian region, the famous $\dfrac{1}{\pi}\arctan\dfrac{\pi}{L}$. After that in 1995 Jones and Solovtsov discovered the coupling which appears to be finite for all s and coincides with Radyushkin one for $s \geq \Lambda^2$, namely $\mathfrak{A}_1[L]$ in Eq. (2b). Just in the same time Beneke et al. [4, 5], using the renormalon based approach, and Shirkov and Solovtsov [6], using the same dispersion approach of [1], discovered ghost-free coupling $\mathcal{A}_1[L]$, Eq. (2a), in Euclidean region.

But Shirkov–Solovtsov approach, named APT, was more powerful: in Euclidean domain, $-q^2 = Q^2$, $L = \ln Q^2/\Lambda^2$, it generates the following set of images for the effective coupling and its n-th powers, $\{\mathcal{A}_n[L]\}_{n\in\mathbb{N}}$, whereas in Minkowskian domain, $q^2 = s$, $L_s = \ln s/\Lambda^2$, it generates another set, $\{\mathfrak{A}_n[L_s]\}_{n\in\mathbb{N}}$ (see also in [7]). APT is based on the RG and causality that guaranties standard perturbative UV asymptotics and spectral properties. Power series $\sum_m d_m a_s^m[L]$ transforms into non-power series $\sum_m d_m \mathcal{A}_m[L]$ in APT.

By the analytization in APT for an observable $f(Q^2)$ we mean the "Källen–Lehman" representation

$$\left[f(Q^2)\right]_{\text{an}} = \int_0^\infty \frac{\rho_f(\sigma)}{\sigma + Q^2 - i\epsilon}\, d\sigma \tag{1}$$

with $\rho_f(\sigma) = \dfrac{1}{\pi}\mathbf{Im}\left[f(-\sigma)\right]$. Then in the one-loop approximation $\rho_1(\sigma) = 1/\sqrt{L_\sigma^2 + \pi^2}$ and

$$\mathcal{A}_1[L] = \int_0^\infty \frac{\rho_1(\sigma)}{\sigma + Q^2}\, d\sigma = \frac{1}{L} - \frac{1}{e^L - 1}\,, \tag{2a}$$

$$\mathfrak{A}_1[L_s] = \int_s^\infty \frac{\rho_1(\sigma)}{\sigma}\, d\sigma = \frac{1}{\pi}\arccos\frac{L_s}{\sqrt{\pi^2 + L_s^2}}\,, \tag{2b}$$

whereas analytic images of the higher powers ($n \geq 2, n \in \mathbb{N}$) are:

$$\begin{pmatrix}\mathcal{A}_n[L] \\ \mathfrak{A}_n[L_s]\end{pmatrix} = \frac{1}{(n-1)!}\left(-\frac{d}{dL}\right)^{n-1}\begin{pmatrix}\mathcal{A}_1[L] \\ \mathfrak{A}_1[L_s]\end{pmatrix}. \tag{3}$$

At first glance, the APT is a complete theory providing tools to produce an analytic answer for any perturbative series in QCD. But in 2001 Karanikas and Stefanis [8] suggested the principle of analytization "as a whole" in the Q^2 plane for hadronic observables, calculated perturbatively. More precisely, they proposed the analytization recipe for terms like $\int_0^1 dx \int_0^1 dy\, \alpha_s\left(Q^2 xy\right) f(x) f(y)$, which can be treated as an effective account for the logarithmic terms in the next-to-leading-order approximation of the perturbative QCD. This actually generalizes the analytic approach suggested in [9]. Indeed, in the standard QCD PT one has also:
(i) the factorization procedure in QCD that gives rise to the appearance of logarithmic factors of the type: $a_s^\nu[L]\, L$;
(ii) the RG evolution that generates evolution factors of the type: $B(Q^2) = \left[Z(Q^2)/Z(\mu^2)\right]\, B(\mu^2)$, which reduce in the one-loop approximation to $Z(Q^2) \sim a_s^\nu[L]$ with

[1] We use notations $f(Q^2)$ and $f[L]$ in order to specify the arguments we mean — squared momentum Q^2 or its logarithm $L = \ln(Q^2/\Lambda^2)$, that is $f[L] = f(\Lambda^2 \cdot e^L)$ and Λ^2 is usually referred to $N_f = 3$ region.

$\nu=\gamma_0/(2b_0)$ being a fractional number.
All that means that in order to generalize APT in the "analytization as a whole" direction one needs to construct analytic images of new functions: a_s^ν, $a_s^\nu L^m, \ldots$. This task has been performed in the frames of the so-called FAPT, suggested in [10,11]. Now we briefly describe this approach.

In the one-loop approximation using recursive relation (3) we can obtain explicit expressions for $\mathcal{A}_\nu[L]$ and $\mathfrak{A}_\nu[L]$:

$$\mathcal{A}_\nu[L]=\frac{1}{L^\nu}-\frac{F(e^{-L},1-\nu)}{\Gamma(\nu)}\,; \tag{4a}$$

$$\mathfrak{A}_\nu[L]=\frac{\sin\left[(\nu-1)\arccos\left(\frac{L}{\sqrt{\pi^2+L^2}}\right)\right]}{\pi(\nu-1)\left(\pi^2+L^2\right)^{(\nu-1)/2}}\,. \tag{4b}$$

Here $F(z,\nu)$ is reduced Lerch transcendental function, which is an analytic function in ν. They have very interesting properties, which we discussed extensively in our previous papers [10–13].

Construction of FAPT with fixed number of quark flavors, N_f, is a two-step procedure: we start with the perturbative result $\left[a_s(Q^2)\right]^\nu$, generate the spectral density $\rho_\nu(\sigma)$ using Eq. (1), and then obtain analytic couplings $\mathcal{A}_\nu[L]$ and $\mathfrak{A}_\nu[L]$ via Eqs. (2). Here N_f is fixed and factorized out. We can proceed in the same manner for N_f-dependent quantities: $\left[\alpha_s(Q^2;N_f)\right]^\nu \Rightarrow \bar{\rho}_\nu(\sigma;N_f)=\bar{\rho}_\nu[L_\sigma;N_f]\equiv\rho_\nu(\sigma)/\beta_f^\nu \Rightarrow \bar{\mathcal{A}}_\nu[L;N_f]$ and $\bar{\mathfrak{A}}_\nu[L;N_f]$ — here N_f is fixed, but not factorized out.

Global version of FAPT [12], which takes into account heavy-quark thresholds, is constructed along the same lines but starting from global perturbative coupling $\left[\alpha_s^{\text{glob}}(Q^2)\right]^\nu$, being a continuous function of Q^2 due to choosing different values of QCD scales Λ_f, corresponding to different values of N_f. We illustrate here the case of only one heavy-quark threshold at $s=m_4^2$, corresponding to the transition $N_f=3\to N_f=4$. Then we obtain the discontinuous spectral density

$$\rho_n^{\text{glob}}(\sigma)=\theta\left(L_\sigma<L_4\right)\,\bar{\rho}_n\left[L_\sigma;3\right] + \theta\left(L_4\leq L_\sigma\right)\,\bar{\rho}_n\left[L_\sigma+\lambda_4;4\right]\,, \tag{5}$$

with $L_\sigma\equiv\ln\left(\sigma/\Lambda_3^2\right)$, $L_f\equiv\ln\left(m_f^2/\Lambda_3^2\right)$ and $\lambda_f\equiv\ln\left(\Lambda_3^2/\Lambda_f^2\right)$ for $f=4$, which is expressed in terms of fixed-flavor spectral densities with 3 and 4 flavors, $\bar{\rho}_n[L;3]$ and $\bar{\rho}_n[L+\lambda_4;4]$. However it generates the continuous Minkowskian coupling

$$\mathfrak{A}_\nu^{\text{glob}}[L]=\theta\left(L<L_4\right)\left(\bar{\mathfrak{A}}_\nu[L;3]+\Delta_{43}\bar{\mathfrak{A}}_\nu\right) + \theta\left(L_4\leq L\right)\,\bar{\mathfrak{A}}_\nu[L+\lambda_4;4]\,. \tag{6a}$$

with $\Delta_{43}\bar{\mathfrak{A}}_\nu=\bar{\mathfrak{A}}_\nu[L_4+\lambda_4;4]-\bar{\mathfrak{A}}_\nu[L_4;3]$ and the analytic Euclidean coupling $\mathcal{A}_\nu^{\text{glob}}[L]$

$$\mathcal{A}_\nu^{\text{glob}}[L]=\bar{\mathcal{A}}_\nu[L+\lambda_4;4] + \int\limits_{-\infty}^{L_4}\frac{\bar{\rho}_\nu[L_\sigma;3]-\bar{\rho}_\nu[L_\sigma+\lambda_4;4]}{1+e^{L-L_\sigma}}\,dL_\sigma \tag{6b}$$

(for more detail see in [12]).

2. Resummation in the one-loop APT and FAPT

We consider now the perturbative expansion of a typical physical quantity, like the Adler function and the ratio R, in the one-loop APT. Due to limited space of our presentation we provide all formulas only for quantities in Minkowski region:

$$\mathcal{R}[L]=\sum_{n=1}^{\infty}d_n\,\mathfrak{A}_n[L]\,. \tag{7}$$

We suggest that there exist the generating function $P(t)$ for coefficients $\tilde{d}_n=d_n/d_1$:

$$\tilde{d}_n=\int_0^\infty P(t)\,t^{n-1}dt\quad\text{with}\quad\int_0^\infty P(t)\,dt=1\,. \tag{8}$$

To shorten our formulae, we use for the integral $\int_0^\infty f(t)P(t)dt$ the following notation: $\langle\langle f(t)\rangle\rangle_{P(t)}$. Then coefficients $d_n=d_1\,\langle\langle t^{n-1}\rangle\rangle_{P(t)}$ and as has been shown in [14] we have the exact result for the sum in (7)

$$\mathcal{R}[L]=d_1\,\langle\langle\mathfrak{A}_1[L-t]\rangle\rangle_{P(t)}\,. \tag{9}$$

The integral in variable t here has a rigorous meaning, ensured by the finiteness of the coupling $\mathfrak{A}_1[t]\leq 1$ and fast fall-off of the generating function $P(t)$.

In our previous publications [12, 15] we have constructed generalizations of these results, first, to the case of the global APT, when heavy-quark thresholds are taken into account. Then one starts with the series of the type (7), where $\mathfrak{A}_n[L]$ are substituted by their global analogs $\mathfrak{A}_n^{\text{glob}}[L]$ (note that due to different normalizations of global couplings, $\mathfrak{A}_n^{\text{glob}}[L]\simeq\mathfrak{A}_n[L]/\beta_f$, the coefficients d_n should be also changed). Then

$$\mathcal{R}^{\text{glob}}[L]=d_1\theta(L<L_4)\langle\langle\Delta_4\bar{\mathfrak{A}}_1[t]+\bar{\mathfrak{A}}_1\Big[L-\frac{t}{\beta_3};3\Big]\rangle\rangle_{P(t)} + d_1\theta(L\geq L_4)\langle\langle\bar{\mathfrak{A}}_1\Big[L+\lambda_4-\frac{t}{\beta_4};4\Big]\rangle\rangle_{P(t)}\,; \tag{10}$$

where $\Delta_4\bar{\mathfrak{A}}_\nu[t]\equiv\bar{\mathfrak{A}}_\nu\Big[L_4+\lambda_4-t/\beta_4;4\Big]-\bar{\mathfrak{A}}_\nu\Big[L_3-t/\beta_3;3\Big]$.

The second generalization has been obtained for the case of the global FAPT. Then the starting point is the series of the type $\sum_{n=0}^{\infty} d_n\, \mathfrak{A}^{\rm glob}_{n+\nu}[L]$ and the result of summation is a complete analog of Eq. (10) with substitutions

$$P(t) \Rightarrow P_\nu(t) = \int_0^1 P\left(\frac{t}{1-x}\right)\frac{\nu\, x^{\nu-1}dx}{1-x}\,, \qquad (11)$$

$d_0 \Rightarrow d_0\,\bar{\mathfrak{A}}_\nu[L]$, $\bar{\mathfrak{A}}_1[L-t] \Rightarrow \bar{\mathfrak{A}}_{1+\nu}[L-t]$, and $\Delta_4\bar{\mathfrak{A}}_1[t] \Rightarrow \Delta_4\bar{\mathfrak{A}}_{1+\nu}[t]$. All needed formulas have been also obtained in parallel for the Euclidean case.

3. Applications to Higgs boson decay

Here we analyze the Higgs boson decay to a $\bar{b}b$ pair. For its width we have

$$\Gamma(\mathrm{H}\to b\bar{b}) = \frac{G_F}{4\sqrt{2}\pi}\, M_H\, \widetilde{R}_{\rm s}(M_H^2) \qquad (12)$$

with $\widetilde{R}_{\rm s}(M_H^2) \equiv m_b^2(M_H^2)\, R_{\rm s}(M_H^2)$ and $R_{\rm s}(s)$ is the R-ratio for the scalar correlator, see for details in [10, 16]. In the one-loop FAPT this generates the following non-power expansion[2]:

$$\widetilde{\mathcal{R}}_{\rm s}[L] = 3\,\hat{m}^2_{(1)}\left\{\mathfrak{A}^{\rm glob}_{\nu_0}[L] + d_1^{\rm s}\sum_{n\geq 1}\frac{\tilde{d}_n^{\rm s}}{\pi^n}\,\mathfrak{A}^{\rm glob}_{n+\nu_0}[L]\right\}, \qquad (13)$$

where $\hat{m}^2_{(1)} = 9.05 \pm 0.09$ GeV^2 is the RG-invariant of the one-loop $m_b^2(\mu^2)$ evolution $m_b^2(Q^2) = \hat{m}^2_{(1)}\,\alpha_s^{\nu_0}(Q^2)$ with $\nu_0 = 2\gamma_0/b_0(5) = 1.04$ and γ_0 is the quark-mass anomalous dimension. This value $\hat{m}^2_{(1)}$ has been obtained using the one-loop relation [17] between the pole b-quark mass of [18] and the mass $m_b(m_b)$.

We take for the generating function $P(t)$ the Lipatov-like model of [15] with $\{c = 2.4,\ \beta = -0.52\}$

$$\tilde{d}_n^{\rm s} = c^{n-1}\,\frac{\Gamma(n+1) + \beta\,\Gamma(n)}{1+\beta}\,, \qquad (14a)$$

$$P_{\rm s}(t) = \frac{(t/c)+\beta}{c\,(1+\beta)}\, e^{-t/c}\,. \qquad (14b)$$

It gives a very good prediction for $\tilde{d}_n^{\rm s}$ with $n = 2, 3, 4$, calculated in the QCD PT [16]: 7.50, 61.1, and 625 in comparison with 7.42, 62.3, and 620. Then we apply FAPT resummation technique to estimate how good is FAPT in approximating the whole sum $\widetilde{\mathcal{R}}_{\rm s}[L]$ in the range $L \in [11.5, 13.7]$ which corresponds to the range $M_H \in [60, 180]$ GeV^2 with $\Lambda^{N_f=3}_{\rm QCD} = 189$ MeV and $\mathfrak{A}_1^{\rm glob}(m_Z^2) = 0.122$. In this range we have ($L_6 = \ln(m_t^2/\Lambda_3^2)$)

$$\frac{\widetilde{\mathcal{R}}_{\rm s}[L]}{3\,\hat{m}^2_{(1)}} = \mathfrak{A}^{\rm glob}_{\nu_0}[L] + \frac{d_1^{\rm s}}{\pi}\,\langle\langle\bar{\mathfrak{A}}_{1+\nu_0}\Big[L+\lambda_5-\frac{t}{\pi\beta_5};5\Big]\rangle\rangle_{P^{\rm s}_{\nu_0}} + \frac{d_1^{\rm s}}{\pi}\,\langle\langle\Delta_6\bar{\mathfrak{A}}_{1+\nu_0}\left[\frac{t}{\pi}\right]\rangle\rangle_{P^{\rm s}_{\nu_0}} \qquad (15)$$

with $P^{\rm s}_{\nu_0}(t)$ defined via Eqs. (14) and (11).

Now we analyze the accuracy of the truncated FAPT expressions

$$\widetilde{\mathcal{R}}_{\rm s}[L;N] = 3\,\hat{m}^2_{(1)}\left[\mathfrak{A}^{\rm glob}_{\nu_0}[L] + d_1^{\rm s}\sum_{n=1}^{N}\frac{\tilde{d}_n^{\rm s}}{\pi^n}\,\mathfrak{A}^{\rm glob}_{n+\nu_0}[L]\right] \qquad (16)$$

and compare them with the total sum $\widetilde{\mathcal{R}}_{\rm s}[L]$ in Eq. (15) using relative errors $\Delta_N[L] = 1 - \widetilde{\mathcal{R}}_{\rm s}[L;N]/\widetilde{\mathcal{R}}_{\rm s}[L]$. In Fig. 1 we show these errors for $N = 2$, $N = 3$, and $N = 4$ in the analyzed range of $L \in [11, 13.8]$. We see that already $\widetilde{\mathcal{R}}_{\rm s}[L;2]$ gives accuracy of the order of 2.5%, whereas $\widetilde{\mathcal{R}}_{\rm s}[L;3]$ of the order of 1%.

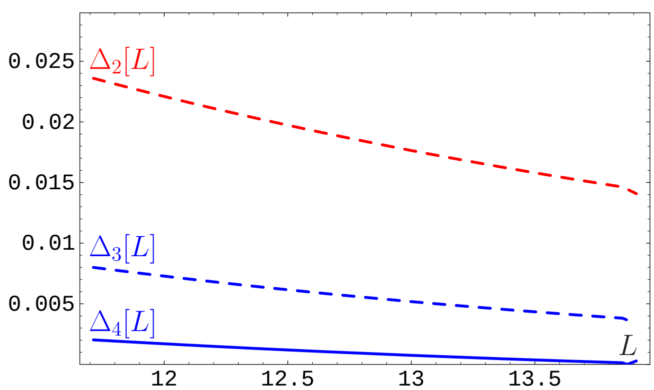

Fig. 1. The relative errors $\Delta_N[L]$, $N = 2, 3$ and 4, of the truncated FAPT in comparison with the exact summation result, Eq. (15).

Looking in Fig. 1 we understand that only in order to have the accuracy better than 0.5% we need to take into account the 4-th correction. We verified also that the uncertainty due to $P(t)$-modelling is small $\lesssim 0.6\%$, while the on-shell mass uncertainty is of the order of 2%. The overall uncertainty then is of the order of 3%, see in Fig. 2.

[2] Appearance of denominators π^n in association with the coefficients $\tilde{d}_n$ is due to d_n normalization.

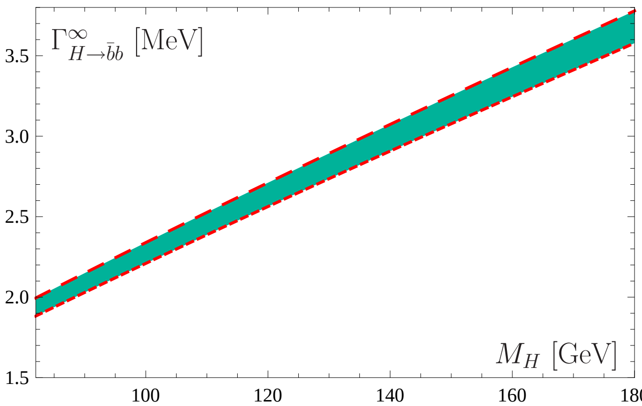

Fig. 2. The width $\Gamma_{H\to b\bar{b}}$ as a function of the Higgs boson mass M_H in the resummed FAPT. The width of the shaded strip is due to the overall uncertainties, induced by the uncertainties of the resummation procedure and the pole mass error-bars.

4. Conclusions

In this report we described the resummation approach in the global versions of the one-loop APT and FAPT and argued that it produces finite answers, provided the generating function $P(t)$ of perturbative coefficients d_n is known. The main conclusion is: To achieve an accuracy of the order of 1% it is enough to take into account up to the third correction—in complete agreement with Kataev&Kim [17]. The d_4 coefficient value is needed only to estimate corresponding generating functions $P(t)$.

Acknowledgements

This work was supported in part by the Russian Foundation for Fundamental Research, grants No. № 07-02-91557 and 08-01-00686, the BRFBR–JINR Cooperation Programme, contract No. F08D-001, and the Heisenberg–Landau Programme under grant 2009.

1. N. N. Bogolyubov, A. A. Logunov, and D. V. Shirkov, Soviet Physics JETP **10** (1960) 574.
2. A. V. Radyushkin, JINR Rapid Commun. **78** (1996) 96; [JINR Preprint, E2-82-159, 26 Febr. 1982; arXiv: hep-ph/9907228].
3. N. V. Krasnikov and A. A. Pivovarov, Phys. Lett. **B116** (1982) 168.
4. M. Beneke and V. M. Braun, Phys. Lett. **B348** (1995) 513.
5. P. Ball, M. Beneke, and V. M. Braun, Nucl. Phys. **B452** (1995) 563.
6. D. V. Shirkov and I. L. Solovtsov, JINR Rapid Commun. **2** [76] (1996) 5; Phys. Rev. Lett. **79** (1997) 1209; Theor. Math. Phys. **150** (2007) 132.
7. Y. A. Simonov, Phys. Atom. Nucl. **65** (2002) 135.
8. A. I. Karanikas and N. G. Stefanis, Phys. Lett. **B504** (2001) 225; **B636**, 330 (2006).
9. N. G. Stefanis, W. Schroers, and H.-C. Kim, Phys. Lett. **B449** (1999) 299; Eur. Phys. J. **C18** (2000) 137.
10. A. P. Bakulev, S. V. Mikhailov, and N. G. Stefanis, Phys. Rev. D**72** (2005) 074014, 119908(E); *ibid.* **75** (2007) 056005; **77** (2008) 079901(E).
11. A. P. Bakulev, A. I. Karanikas, and N. G. Stefanis, Phys. Rev. D**72** (2005) 074015.
12. A. P. Bakulev Phys. Elem. Part. Nucl. **40** (2009) 715.
13. N. G. Stefanis, arXiv:0902.4805 [hep-ph].
14. S. V. Mikhailov, JHEP **06** (2007) 009.
15. A. P. Bakulev and S. V. Mikhailov, in *Proc. Int. Seminar on Contemp. Probl. of Part. Phys., dedicated to the memory of I. L. Solovtsov, Dubna, Jan. 17–18, 2008.*, Eds. A. P. Bakulev *et al.* (JINR, Dubna, 2008), pp. 119–133 (arXiv:0803.3013).
16. P. A. Baikov, K. G. Chetyrkin, and J. H. Kühn, Phys. Rev. Lett. **96** (2006) 012003.
17. A. L. Kataev and V. T. Kim, in *Proc. Int. Seminar on Contemp. Probl. of Part. Phys., dedicated to the memory of I. L. Solovtsov, Dubna, Jan. 17–18, 2003.*, Eds. A. P. Bakulev *et al.* (JINR, Dubna, 2008), pp. 167–182 (arXiv:0804.3992); plenary tak, presented by A.L.K at XII Int. Workshop on Advanced Computing and Analysis Techniques in Physics Research, November 3–7, 2008, Erice, Sicily, Italy (arXiv:0902.1442).
18. J. H. Kuhn and M. Steinhauser, Nucl. Phys. **B619** (2001) 588.